STRUCTURAL MEMBRANE APPROACH FOR THE DESIGN OF SHELLS

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ABSTRACT

The structural membrane approach is a powerful tool in the analysis of flat plates and thin shells. It is based upon the fact that an elastic membrane may be approximated with three basic and easily analyzed shell units. The present paper summarizes the result of these efforts.

1. INTRODUCTION

The structural membrane (SM) approach makes use of three unique shapes: the elliptical dome (D), the hyperbolic paraboloid (HP) and the elliptical logarithmical funnel (F) by fusing these into smooth continous surfaces. The equations for these shapes may be expressed as follows:

$$z = k_2(x^2-y^2) \quad (1)$$

$$z = k_1(x^2+y^2) \quad (2)$$

$$z = k_3\ln(x/d) \quad (3)$$

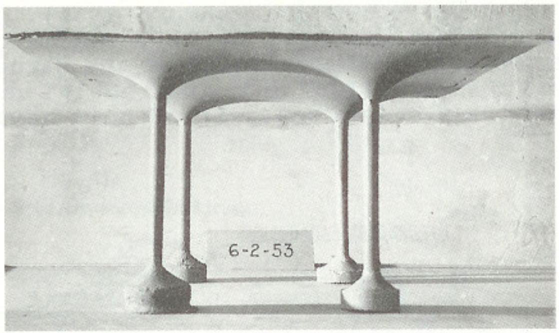

Fig. 1. Elastic Membrane Model

Each of the surfaces satisfy the basic differential equations for bending moments in plates:

$$\frac{\delta^2 M}{\delta x^2} + \frac{\delta^2 M}{\delta y^2} = -w \quad (4)$$

and for the deflection of elastic membranes:

$$\frac{\delta^2 e}{\delta x^2} + \frac{\delta^2 e}{\delta y^2} = -w \quad (5)$$

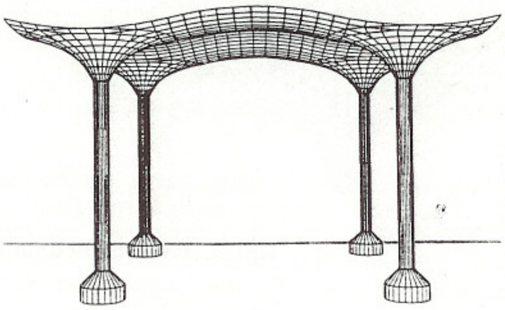

Fig. 2. Structural Membrane Model

where M is the bending moments in the plate and e is the deflected shape of the elastic membrane and w is the uniform vertical load.

The structural membrane shells are arranged according to a series of general rules, resulting in a continuous shell that in appearance and behavior is very close to the similarly supported elastic membrane.

When compared to other methods of analysis, such as the finite element method, the structural membrane approach is less time consuming and the answer less obscured by heavy number crunching.

The basic equations are the same for both plates and shells. In this paper only their application to thin shells is discussed. A series of examples has been shown in a companion paper.

The elastic membrane, by its nature, is the optimum shape of a thin shell carrying a uniform load from column to column. The structural membrane, as a close approximation to this elastic membrane, furnishes a practical and effective form of thin shells to span between regularly or randomly spaced columns.

The basic structural membrane approach, which pointed out the similarity between the structural membrane and the elastic membrane, was published some time ago (Saether 1963). The similarity in form of the differential equations of the elastic membrane deflections and the bending moments in flat plates has long been known (Timeshenko 1959). A combination of the two principles was used by this author (Saether 94). This paper demonstrated the validity of the Structural Membrane Theory used in plate design and indirectly validated its use for the formulation and analysis of thin shells.

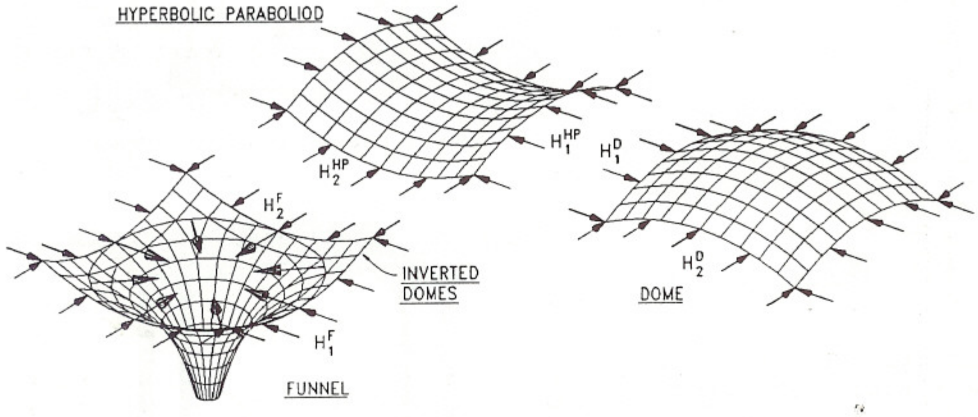

Fig. 3. Basic Structural Membrane Units

2. THE STRUCTURAL MEMBRANE

Several separate incidents led up to the discovery of the structural membrane. Studies of various shell designs, some plain looking shells, such as various forms of barrel shells, required complicated analytical approaches. Other relatively complex looking shapes such as domes, hyperbolic paraboloids and funnels as shown in Fig. 3 were readily analyzed. It could furthermore be shown that it was possible to fuse these three shapes into one continuous, smooth flowing surface while still complying with all equilibrium requirements. This combined surface is what is referred to as a Structural Membrane. To arrive at this combined surface, the relative arrangement of the three surfaces, as well as the exact location of the transition of one shape to the other, were dictated by a set of distinct rules:

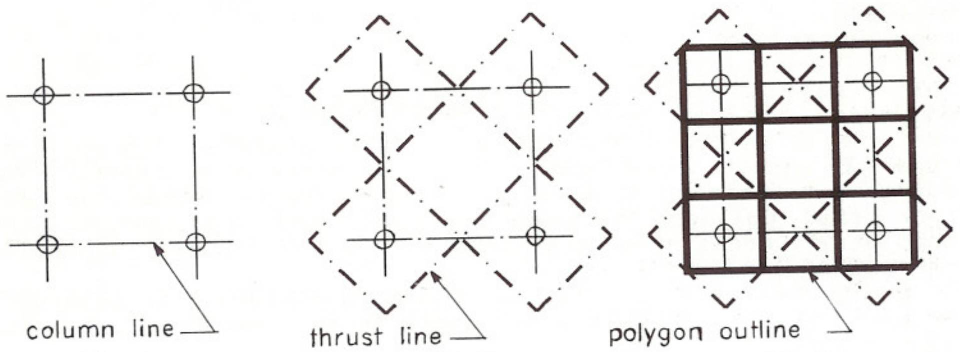

Fig. 4a. Fig. 4b. Fig. 4c.
Construction of Square SM Layout

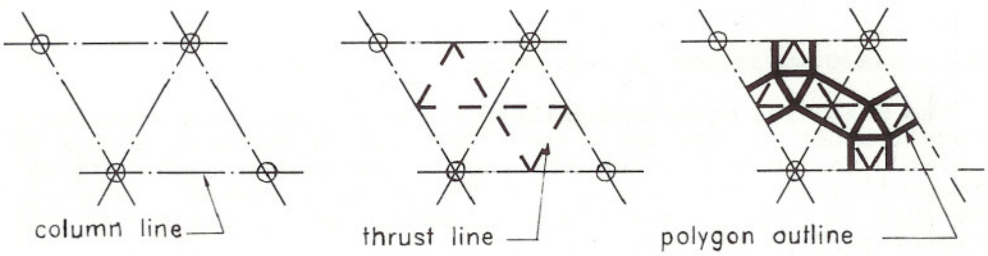

Fig. 5a. Fig. 5b. Fig. 5c.
Construction of Triangular SM Layout

1. All columns must be connected to a grid of triangular areas. The lines are referred to as "column lines," (see Figs. 4a and 5a). Except for the angle facing an exterior edge, no angle in a triangle thus created may be greater than 70°. Instead in these cases, polygons with four or more sides are allowed within the grid.
2. The midpoints of the column lines must be connected by straight lines that are referred to as thrust lines. Any given

thrust line can bypass only one column, (see Figs. 4b and 5b).
3. The midpoints of the thrust lines must be connected by straight lines. Each new line may bypass only one column line midpoint. The latter lines outline the basic shell polygons in the structural membrane, (see Figs. 4c and 5c).
4. To identify the resulting polygons as to their SM definitions, the following rules apply: The polygons free of thrust lines and columns describe the domes. The polygons containing columns describe the funnel outlines. The four-sided polygons between the domes and the funnels, and completely containing the thrust lines, are the hypebolic paraboloids, (see Fig. 6).

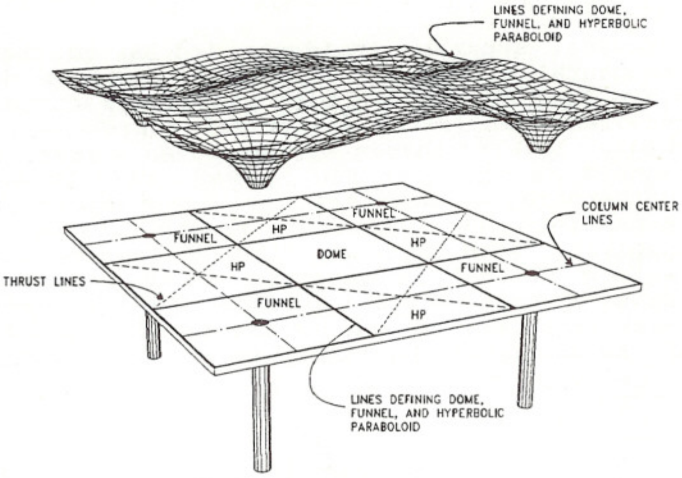

Fig. 6. Square SM Shell Unit Regions

The square layout in Fig. 4a is not subdivided into triangles as this would have yielded one angle in the resulting triangles larger than 70^0.

In Fig. 5a the parallelograms have been subdivided into triangles as the resulting angles are all smaller than 70^0.

3. BASIC EQUATIONS

The following equations satisfy all equilibrium and continuity requirements in the SM units. The superscripts D, F and HP refer to domes, funnels and hyperbolic paraboloids, respectively. The geometry of the SM Basic Element (SMBE) is defined in Figs. 7 and 8 and is controlled by the use of the variables (a,b,c and d).

The span between the center of one valley (HP) across the dome to the next valley (HP) is:

$$L^{D/HP} = (a_1 + b_1) + (a_2 + b_2) \qquad (6a)$$

where a_1 and b_1 extend from the center of the dome to the center of the supporting HP; and a_2 and b_2 extend to the center of the opposite HP. This is shown in Fig. 8. For symmetrical layouts the span is:

$$L^{D}/HP = 2(a+b) \qquad (6b)$$

To start a layout of a SM shell, the dome height h^D is selected by the engineer. The ratio of a/b is referred to as $n^{D/HP}$. From the continuity requirements, the ratio of the dome height h^D to the depth of the hyperbolic paraboloid h^{HP} is found to be equal to the same $n^{D/HP}$:

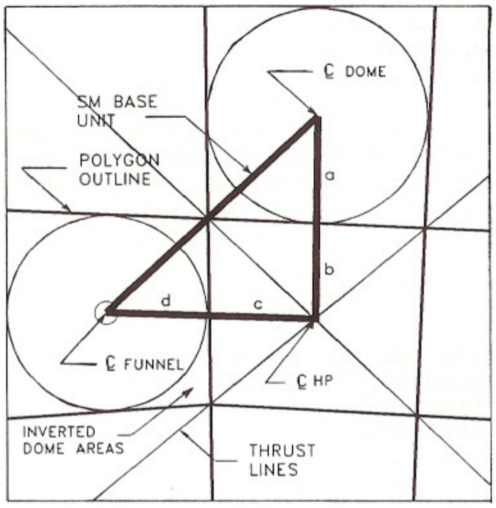

Fig. 7. SM Basic Element

$$\frac{h^D}{h^{HP}} = \frac{a}{b} = n^{D/HP} \qquad (7)$$

From the equilibrium requirements, and by assuming that for any circular dome the exterior load w is divided equally in any two orthogonal directions, the following formulas are derived:

The thrust H_1^D in the dome is:

$$H_1^D = \frac{wa^2}{4h_D} \qquad (8)$$

The depth of the hyperbolic paraboloid h^{HP} is calculated as:

$$h^{HP} = \frac{h^D}{n^{D/HP}} \qquad (9)$$

The span from one column along the valley (HP) to the next column is:

$$L^{HP/F} = (c_1 + d_1) + (c_2 + d_2) \qquad (10)$$

where $c_1 + d_1$ is the distance from center of HP to center of one funnel (center of column) and $c_2 + d_2$ is the distance to the center of other funnel (center of the other column) as shown in Fig. 8. It is convenient to borrow the terms *column-strip* and *middle-strip* from flat plates design. (see Fig. 8). From the equilibrium requirements it is found that the thrust across the HP must be:

$$H_1^{HP} = H_1^D \qquad (11)$$

This thrust generates a downward reaction w_1^{HP} within the HP of the magnitude:

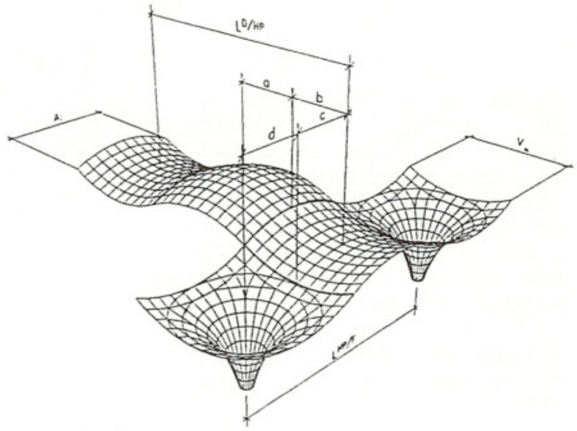

Fig. 8. Middle-Strip/Column-Strip Intersection

$$w_1^{HP} = + \frac{H_1^{HP} h^D \, 2}{b^2 n^{D/HP}} = + \frac{2 H_1^{HP} h^{HP}}{b^2} \qquad (12)$$

The orthogonal thrust H_2^{HP} in the HP, which is the same as the thrust H_1^F along the upper rim of the funnel unit along the top of the inverted domes, must carry this load in addition to the given uniform load w and is expressed as:

$$H_2^{HP} = H_1^F = \frac{(w_1^{HP} + w) \cdot (c)^2}{2h^{HP}} \qquad (13)$$

Again the factor c is one of the basic factors in the SMBE as defined in Figs. 7 and 8.

The outline of the total contributory area, A_{TL}, as carried by any one column, is identified by connecting the surrounding domes with straight lines through the HP centers. For a number of regular layouts these have been calculated and presented in Table 1.

The total load on a column is calculated as:

$$W_{TL} = wA_{TL} \qquad (14)$$

From this the required thrust H_2^F along the rim of the circular funnel is determined as:

$$H_2^F = \frac{w(A_{TL} - \pi d^2)\,c}{4\pi\ dh^{HP}} \tag{15}$$

where c and d are defined in Figs. 7 and 8. The ratio of these two thrust values, one along the rim of the circular funnel H_1^F and one along the top of the invert domes H_2^F, is given by the following expression:

$$RN = \frac{H_2^F}{H_1^F} = \frac{(A_{TL} - \pi d^2)}{2c\pi d(w_1^{HP}/w + 1)} \tag{16}$$

where H_1^F is the thrust along the common border with the HP and H_2^F is the radial thrust at the funnel rim.

The value of RN is found to be always larger than one, which can be calculated for any structural membrane layout. For some common conditions this value is presented in Table 1.

Column layout (1)	m (2)	RN (3)	Contributory area (4)
Hexagon	3	1.70	$3(a + b)(c + d)$
Square	4	1.37	$4(a + b)(c + d)$
Triangular	6	1.24	$6(a + b)(c + d)$
Free-form	Eq. (15)	Eq. (16)	Eq. (17)

Table 1. Basic Factors in Structural Membrane

If the horizontal thrust for circular funnels is specified to be a constant, the corresponding differential equation for the funnel shape becomes:

$$\frac{dz}{dx} = \frac{(A_{TL}-\pi x^2)\,w}{2\pi x H^F} \tag{17}$$

for which the solution is:

$$Z = A\ln(x/d) - B(x^2 - d^2) - h^{HP} \tag{18a}$$

Where:

$$A = A_{TL}w/2\pi H^F \text{ and } B = w/4H^F \tag{18b}$$

With the above equations the SM variable heights e and the horizontal thrusts H in all units are determined.

In all SM shells, the required thrust is calculated along the previously identified thrust lines. The total required tensile force acting along these lines is the sum of the unit thrusts in the shell, starting at the center of one dome on one side and proceeding to the center of the dome on the other side of the same thrust line.

4. CONCLUSIONS

The basic development of the structural membrane has been outlined. Conceptually simple, the structural membrane is an accurate and effective tool for the determination of economical column supported thin shells. This avoids the need for time consuming and costly finite element analysis and can be implemented with hand calculations or incorporated in simple stand-alone computer programs.

APPENDIX I. REFERENCES

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