At the same time the advantages are significant. Using very little concrete and steel, column spacings can be much greater than with most flat shapes. Regardless of spacing, the use of material is reduced. The beauty of the ceiling and architectural interest is increased. Combined window panels under the dome unit provide extremely interesting lighting and ventilation. Being under compression, little or no waterproofing should be required.

The structural membrane is almost ideally shaped from a structural standpoint. Its simplicity in design and construction makes it attractive even in an area where the labor cost is the major economical factor. With its pleasing lines it is felt that it might add beauty to an era geared principally for economy and production.

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Title No. 57-41

The Structural Membrane

By KOLBJORN SAETHER

Even though ideal from a structural point of view, elastic membranes have been almost excluded from the structural field due to the complexity of mathematical work involved. Structural membranes, however, which are close to identical in appearance and structural behavior, permit the use of only elementary mathematics for defining the shape and analyzing the stresses within its surface. The basic theory of funicular shapes and the transformation of these into structural membranes is shown.

Savings on materials typical for all thin shell structures, together with ease of design and construction, are some of the advantages promised by structural membranes.

■ IN THE STRUCTURAL FIELD, where man is mastering larger forces than in almost any other technical branch, applications of the elastic membrane have been limited. Except for steel cables in funicular structures, few available structural materials come even close to having the qualities of an elastic membrane. In the following it will be shown how a stiff material such as concrete may be used with the same efficiency and elegance to carry floor and roof loads as the sail is used to contain the driving force of the wind.

If a complete stress reversal is assumed the shape of an elastic membrane under tension would be ideally suited for a concrete structure under compression. This idea in itself is not new and a vast amount of literature exists on this subject. The problem in the past has been that the funicular shape (the shape of an elastic membrane) is so complex as to obstruct easy assignment of engineering dimensions, stresses, loads, and other necessary data.

In this paper, however, the funicular shape has been very closely approximated by using three known geometrical shapes, all of them readily defined and each one of them already separately used in structural design. These shapes are the parabolic elliptical dome, the hyperbolic paraboloid, and the logarithmic elliptical funnel.

A structural membrane is defined as a combination of two or more of these three geometrical shapes fused together into one continuous surface.

To understand the behavior of a structural membrane it is necessary to examine what takes place when the shape of an elastic membrane is used in a concrete structure under compression. After this it will be shown how the shape of an elastic membrane can be approximated and simplified by the structural membrane.

CONVERTING AN ELASTIC MEMBRANE UNDER TENSION TO A SIMILAR FUNICULAR STRUCTURE UNDER COMPRESSION

By comparing an elastic membrane under tension with the corresponding structural shape under compression a number of conclusions about the structural conditions in the latter shape may be made. Under distributed loads w normal to the membrane, the elastic membrane will deflect in such a manner that the internal stresses S, all acting in the direction of the deflected surface, will have their resultants r in perfect equilibrium with the loads (Fig. 1).

This condition may be expressed as follows:

$$S_{xy} = F_{xy}(w) = F_{xy}(r) \quad \ldots\ldots\ldots\ldots (1)$$

Due to the nature of the elastic membrane the stresses S are all tensile stresses.

If all signs in Eq. (1) are changed, providing the shape is maintained, the equilibrium is not disturbed (Fig. 2). This analogous condition may be expressed mathematically:

$$-S_{xy} = F_{xy}(-w) = F_{xy}(-r) \quad \ldots\ldots\ldots\ldots (2)$$

Whereas Eq. (1) indicates a statical condition that is always satisfied in an elastic membrane, a quick check of Eq. (2) shows that these analogous conditions could not all be handled by the same membrane.

TABLE 1—TRANSFORMATION OF PROPERTIES FROM ELASTIC TO STRUCTURAL MEMBRANE

Elastic membrane [Eq. (1)]	Funicular Structural Shape [Eq. (2)]
Loads on the elastic membrane model to be selected so that they are opposite to the actual loads on the structure	The loads are assumed known as those for which the structure has to be designed (Generally total dead and live load)
Certain tensional edge loads required to stress the membrane and to maintain the proper location of the edges	Complete reversal of all edge loads (Often handled by assuming beam-action along the edges of the structure)
Tension throughout	Compression throughout
Flexible surface	Rigid surface
All bending moments equal zero	All bending moments equal zero
Stabile equilibrium	Labile equilibrium
Shape is automatically that of the deflected membrane	Shape made identical to that of the deflected membrane

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Mr. Saether's experience includes thin shell designs, precast and slip-form construction problems, drilled-in caissons, lift-slab, blast resistant design, and development of a direct design method for prestressed concrete.

Table 1 shows how the properties known to exist in the elastic membrane are transformed in the analogy.

The importance of the analogy given by the transformation from Eq. (1) to Eq. (2) is easily understood by checking each item in the second column of Table 1. Item for item they show the ideal conditions for a concrete shell. The high compressive stresses located within the surface itself permit full use of the compressive strength of the concrete. These stresses are furthermore nearly uniform throughout the shell without concentrated loads or line loads. By definition the structure is initially free of any bending moments. Only secondary moments are introduced by changes in the live load and by edge loads.

Having examined what happens when the exact shape of an elastic membrane is used in a structure under compression, the next step is to define this shape. Instead of doing this mathematically or experimentally, an entirely different approach is shown.

Through observation of elastic membrane models it has been possible in a number of cases to approximate the shape of the elastic membrane with the three specified shapes of a structural membrane. While there is no apparent limit to the number of different elastic membranes which can be approximated in this manner, this paper is confined to the description of three basic types of structural membranes.

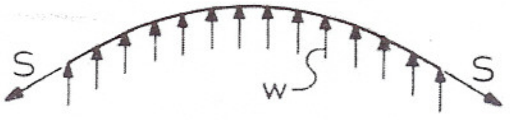

Fig. 1—Equilibrium of loads and stresses in elastic membrane

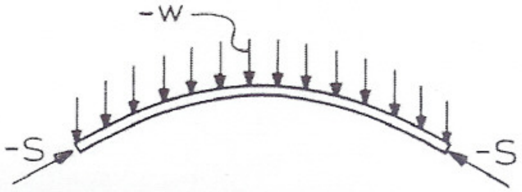

Fig. 2—Equilibrium of loads and stresses in structural membrane

EXAMPLE 1 — AN ELASTIC MEMBRANE SPANNING A ROUND OPENING

The first elastic membrane is created by a rubber sheet placed over a round opening and inflated (Fig. 3a). The resulting shape is found to be a dome (Fig. 3b).

The corresponding structural membrane consists of a circular parabolic dome, a special case of the elliptical parabolic dome. The results in itself being of little importance as this is a well known and commonly used structure, it should be noted that this structural membrane approximates the elastic membrane more completely than does the often used spherical dome.

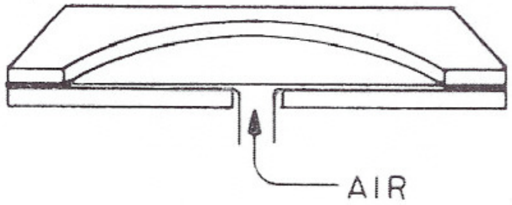

Fig. 3a—Apparatus for creating elastic membrane

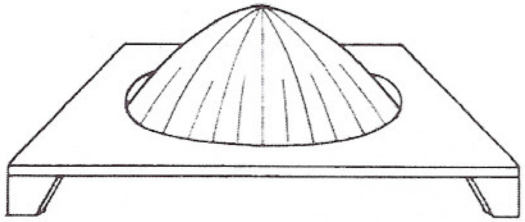

Fig. 3b—Circular parabolic dome

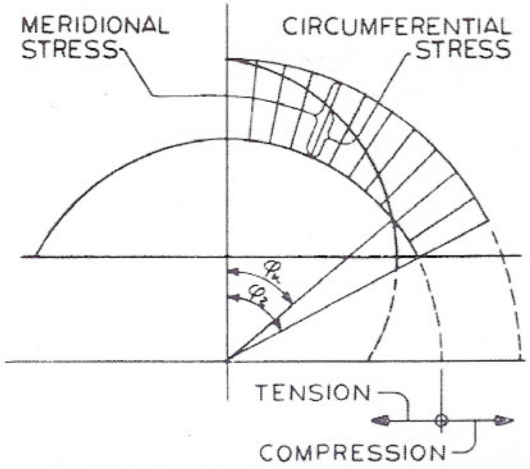

Fig. 4—Stresses in a spherical dome

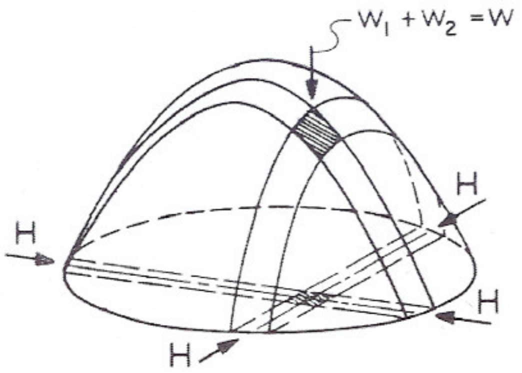

Fig. 5a—Surface generation in parabolic dome

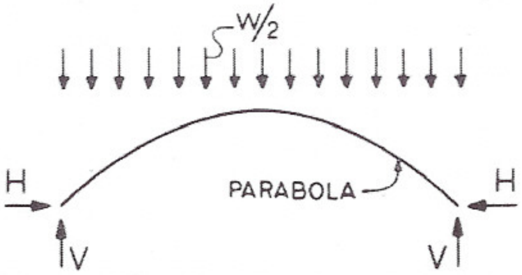

Fig. 5b — Thrust conditions in parabolic dome

The reason for this is apparent by noting that on a spherical dome the circumferential stresses decrease to zero along a definite line below which the dome is in tension in the circumferential direction (Fig. 4). This stress reversal does not exist in the elastic membrane as discussed in connection with Table 1.

The elliptical parabolic dome, on the other hand, is close to identical to the deflected elastic membrane. If a uniformly stretched elastic membrane is deformed only slightly, the stresses in the sheet will undergo only small changes and it may be assumed that the horizontal thrust, *H*, at any point of the sheet will remain close to constant throughout the membrane.

The dome may now be assumed subdivided into two series of parallel running arches intersecting at 90 deg (Fig. 5a). Each element of the dome is covered twice by these arches so that the total unit load on the dome, *W*, may be assumed divided into two parts. W_1 and W_2, the first part carried by one set of parallel running arches, the other part carried by the intersecting set of arches. The following equation will therefore apply to any element on the dome:

$$W_1 + W_2 = W \qquad (3)$$

If it is further assumed that the total load is equally divided between the two sets of arches:

$$W_1 = W_2 = W/2 \qquad (4)$$

and with the assumption that the thrust is constant throughout, the shape of these arches are completely determined, since the only arch that will satisfy these conditions is an arch with a parabolic shape (Fig. 5b).

The shape of the deflected elastic membrane is thereby determined to be identical to that of the elliptical paraboloid. This surface may be created by moving one parabola along and normal to another parabola. In a rotation symmetrical case as this is, the two parabolas are identical and the same surface could have been created by rotating the parabola about its axis. The parabolic dome thus discovered is not new; it has been used for some time. The important thing is that it indicates a complete correspondence between the elastic membrane and the structural membrane.

Even though used in a number of structures the parabolic dome permits little flexibility in design. The difficulty of combining more units within the same structure and its requirement of continuous circular support limits its usefulness. The following two examples are quite different in that respect. They are limited by straight, horizontal lines which make it possible to combine any number of units within the same project.

EXAMPLE 2 — SQUARE ELASTIC MEMBRANE TRANSLATED INTO A STRUCTURAL MEMBRANE

The next elastic membrane was built with a rubber sheet over a square opening. To account for the effect of a uniform dead and live load, air pressure was introduced under the membrane. A downward acting concentrated load was applied in the center as shown in Fig. 6, to correspond to the column reaction.

A study of the resulting shape shows a close to rotation symmetrical funnel shape in the vicinity of the column. Towards the corners the membrane has every characteristic of a saddle shape, similar to the hyperbolic paraboloid. A close investigation will further show that the transition from one shape into the other takes place along diagonal junction lines through the midpoint of each side, denoted with *a* in Fig. 7. The corresponding structural membrane consists of a logarithmic circular conoid with the four corners consisting of hyperbolic paraboloids.

The compressive stresses located within the surface itself permit full use of the compressive strength of the concrete. As in the elastic membrane, these stresses are furthermore nearly uniform throughout the shell without concentrated loads or line loads. By definition the

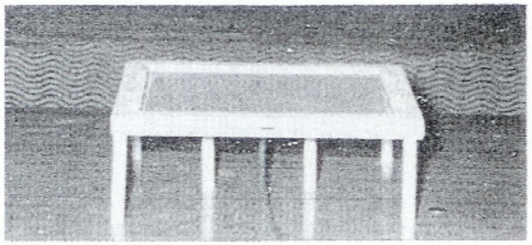

Fig. 6a—Elastic membrane uninflated

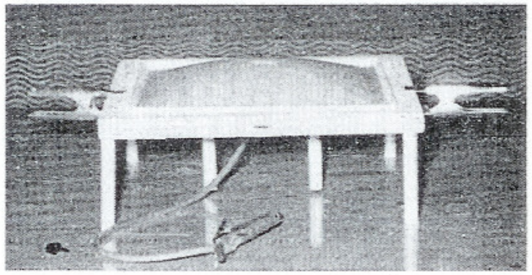

Fig. 6b—Elastic membrane inflated

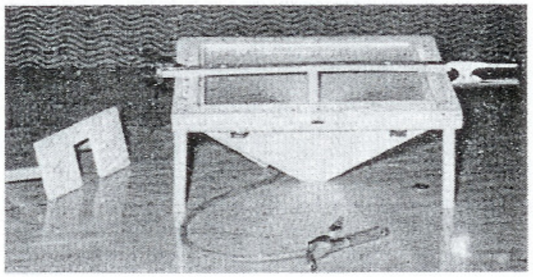

Fig. 6c—Elastic membrane inflated with downward load applied

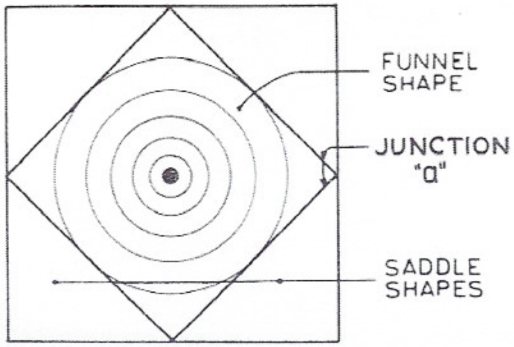

Fig. 7—Shape transition in structural membrane

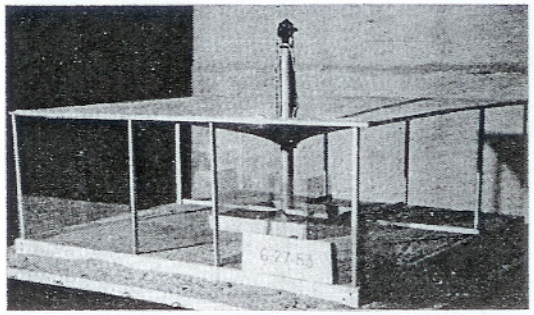

Fig. 8a—One-story one-column house

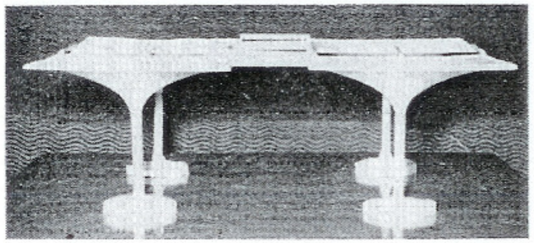

Fig. 8b—Four-column office

structure is initially free of any bending moments. Only secondary moments are introduced by changes in the live load and by removal of the edge loads.

The problem of physically measuring or analyzing a funicular shape is eliminated, and yet the shape offers many of the structural advantages of the funicular shape. This shape can be constructed virtually without beams, and all the concrete, except at the base of the conoid at the column, may be of constant thickness. The models in Fig. 8 show a one-story, one-column home and a four-column office layout with intersecting strips to provide light and ventilation.

The fact that this conoid and hyperbolic paraboloid may be fused into one continuous uninterrupted shape will be shown in detail in Example 3, along with mathematical data establishing and analyzing the use of these shapes.

EXAMPLE 3 — CREATING AND DEFINING A MORE VARIED STRUCTURAL MEMBRANE

An elastic sheet is inflated over a square opening with four downward acting loads. The funicular shape supported by evenly spaced columns is thereby created (Fig. 9a). Air pressure underneath the sheet represents the uniform load on the structure. A thin layer of concrete is cast on top of this sheet, and Fig. 9b presents the resulting shape. The conoid is quickly recognized as the area adjacent to the column with the parabolic dome located diagonally between the columns. These areas have been indicated as Areas 1 and 3, respectively, in Fig. 10. The hyperbolic paraboloid shows up as transition sections between the other two areas and is located half way between the columns along the column lines, denoted as Area 2 in Fig. 10. A close observation of the model further reveals that the transition from one

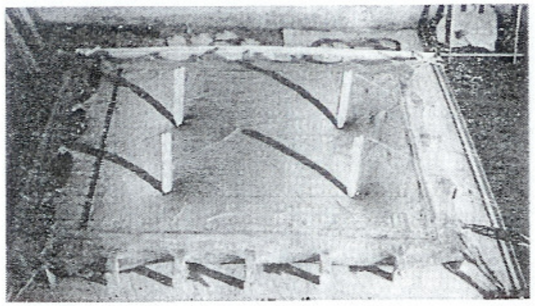

Fig. 9a—Casting funicular shape

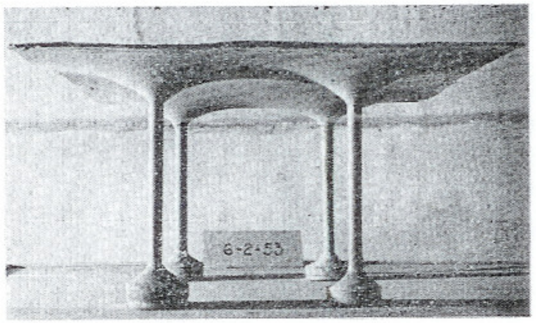

Fig. 9b—Resulting casting

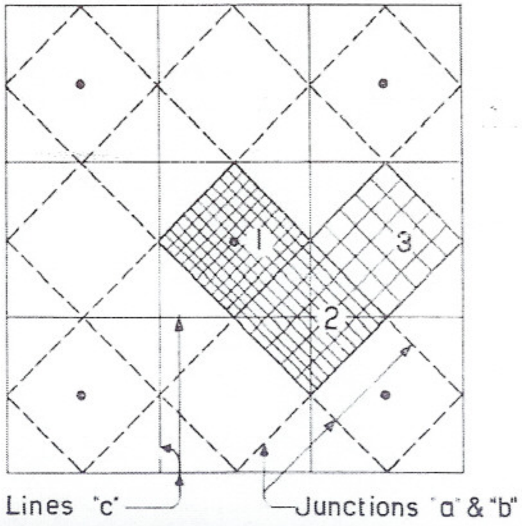

Fig. 10—Plan of structural membrane showing relative location of funnel (1), saddle (2), and dome (3)

area into the adjacent takes place along straight vertical sections, junctions *a* and *b* as shown in the same figure.

In Fig. 11 the same three areas are shown in larger scale. In each area a coordinate system has been introduced. Fig. 12 shows a section through the surface. By selecting the proper parameters the equations for these three areas may be expressed as follows:

Area 1 $$X^2_1 + Y^2_1 = F(Z) \tag{5}$$

Area 1a $$X^2_1 + Y^2_1 = -\frac{L^2}{4}\left(1 - \frac{Z}{h}\right) \tag{5a}$$

Area 2 $$X_2Y_2 = \frac{L^2}{8}\left(\frac{Z}{h}\right) \tag{6}$$

Area 3 $$X^2_3 + Y^2_3 = \frac{L^2}{4}\left(1 - \frac{Z}{h}\right) \tag{7}$$

The three coordinate systems have been used simply for the purpose of keeping the above equations in such shape that they may readily be recognized. The ordinate Z is the same in all cases, the ordinates X and Y differ only in that they have been translated from Area 1 with the amounts $X = +L/2$ and $Y = -L/2$ for Area 2, and $X = +L$ and $Y = 0$ for Area 3, respectively.

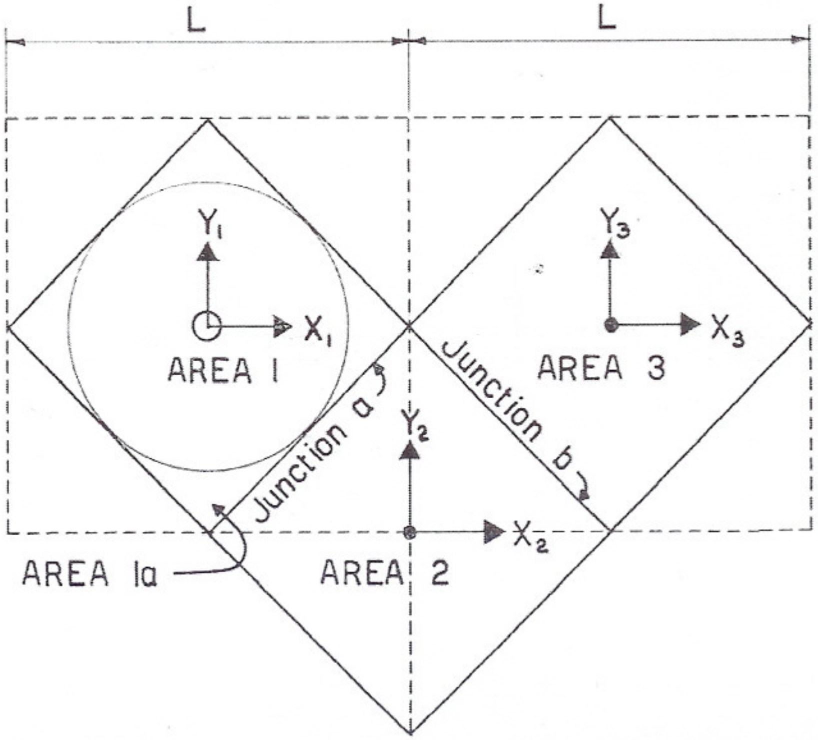

Fig. 11—Expanded view of three areas in Fig. 10

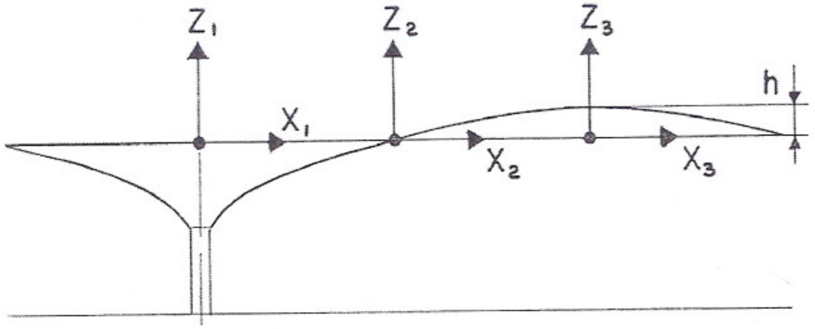

Fig. 12—Section through surface

Area 1 in Fig. 11 has been divided into two parts, one circular Area 1 and four triangular segments denoted as Areas 1a. The above expressions for these areas show that the two Areas 1 and 1a are identical, with 1a as a special case of 1. As it will be shown in the following, Eq. (5) may take several specific shapes. The only limitations are that $F(Z)$ and $F'(Z)$ must coincide with those of Eq. (5a) for values of $X^2_1 + Y^2_1 = L^2/8$, that is: the location of Surface 1 and its slope must be identical with that of Surface 1a along the circle with radius $L/\sqrt{8}$. That these requirements can be satisfied is evident due to the similarity of the two equations. That the other equations describe one continuous shape may not be quite so evident; this will be proven in the following:

Since Eq. (5a) and Eq. (7) are identical but opposite and since Eq. (6) is equal but opposite in the two quadrants limited by the junction lines *a* and *b*, respectively, it is enough to show that Eq. (6) and (7) satisfy the continuity requirements. To establish the location of the surface along the junction the equation for the vertical surface intersecting the structural membrane along the junction line *a* must be written out. Expressed in terms of coordinate System 2 it is:

$$X_2 + Y_2 = \frac{L}{2} \text{ or } X_2 = \frac{L}{2} - Y_2$$

This value for X_2 inserted in Eq. (6) gives the following expression for the junction line:

$$\left(\frac{L}{2} - Y_2\right) Y_2 = \frac{L^2}{8}\left(\frac{Z}{h}\right) \quad \text{or} \quad \frac{L}{2} Y_2 - Y^2_2 = \frac{L^2}{8}\left(\frac{Z}{h}\right)$$

The corresponding vertical surface as expressed in terms of X_3 and Y_3 is:

$$X_3 + Y_3 = -\frac{L}{2} \quad \text{or} \quad X_3 = -\frac{L}{2} - Y_3$$

This value for X_3 inserted in Eq. (7) gives:

$$\left(\frac{L}{2} + Y_3\right)^2 + Y_3 = \frac{L^2}{4}\left(1 - \frac{Z}{h}\right)$$

Since $Y_3 = Y_2 - L/2$ it follows:

$$\left(\frac{L}{2} + Y_2 - \frac{L}{2}\right) + \left(Y_2 - \frac{L}{2}\right)^2 = \frac{L^2}{4}\left(1 - \frac{Z}{h}\right)$$

and

$$2\,Y_2^2 - Y_2 L + \frac{L^2}{4} = \frac{L^2}{4} - \frac{L^2}{4}\,\frac{Z}{h}$$

With $L^2/4$ eliminated and the resulting expression divided by 2 and changing signs throughout it is found:

$$\frac{L}{2}\,Y_2 - Y_2^2 = \frac{L^2}{8}\left(\frac{Z}{h}\right)$$

This is, however, the same curve as established by using Eq. (6) and proves that the two surfaces are at the same elevation at any point along the junction line.

To prove that there is a smooth transition from one surface to the adjacent one, it is enough to prove that the surface is continuous and of constant slope in any one direction outside the direction of the junction line itself. The simplest expressions are obtained by establishing the slope of the surface in the direction parallel to the X-axis. (The Y-axis could have been chosen with the same convenience.) The slope of Area 2 in the direction of the X-axis is expressed by taking the partial derivative of Z in this direction. From Eq. (6) it is:

$$\frac{\partial Z}{\partial X} = \frac{8h}{L^2}\,Y_2$$

and from Eq. (7) the equivalent value is:

$$\frac{\partial Z}{\partial X} = \frac{4h}{L^2}\,2X_3 = \frac{8h}{L^2}\left(-\frac{L}{2} - Y_3\right) = \frac{8h}{L^2}\left(-\frac{L}{2} - Y_2 + \frac{L}{2}\right) = \frac{8h}{L^2}\,(-Y_2)$$

This last expression is equal but opposite to that established by using Eq. (6) and proves that the surface has a constant slope across the junction lines.

The basic geometry of the structural membrane is thus established. Just as there is a series of possible variations to the regular, symmetrical hyperbolic paraboloid the same applies to the structural membrane. Structural layouts where the three basic shapes of the conoid, the hyperbolic paraboloid, and the parabolic dome have been changed from having a regular square outline as shown here to a pattern of triangular and hexagon shapes; all triangular shapes; and all rectangular or all diamond shapes have been successfully combined to cover a long variety of column arrangements.

STRESSES IN STRUCTURAL MEMBRANE SUPPORTED ON EVENLY SPACED COLUMNS DUE TO UNIFORMLY DISTRIBUTED VERTICAL LOAD

Even though partial loading may control the design of some thin shell structures, the great majority of shells constructed till now have been based upon a design considering the uniform total load only. This is partly due to the practical limitation of our knowledge about shell structures, but also due to the fact that most shells, and especially those of double curvature, have remarkable ability to redistribute the load with the effect that the total load governs the design. In this paper only the total, uniformly distributed load will be considered.

If the unit horizontal thrust were known for any one section of a shell the actual stresses within the surface itself could be readily determined by the following expression:

$$F = H \cos w \,/\, \cos u \quad \text{(8)}$$

where H is the unit horizontal thrust, u, the angle of slope of the shell surface in a direction normal to the horizontal projection of the section, and w is the slope of the section itself. The unit stress F would not necessarily be exactly acting normal to the section, but for small values of the angle w the discrepancy would be small and could safely be ignored in the design. For larger values of w an adjustment may be required. Eq. (8) is general and applicable to any type of thin shell structure, and the stress investigation is thereby reduced to the determination of the required horizontal thrust.

As shown in Example 1 the parabolic dome may be assumed to be divided into two series of intersecting arches by vertical planes running parallel with the junction lines. These arches so established will all be identical, a fact readily verified by the knowledge that a parabolic dome may be described by translating one parabola along and normal to a similar parabola. Since the height of these parabolas is $h/2$ and the span of the same parabolas is equal to $L/\sqrt{2}$ and further since the load, due to absolute symmetry of the shell in this area, may be assumed divided equally between the two sets of arches, the required thrust throughout and along the edges of the dome area is determined to be:

$$H_1 = \frac{w\,L^2}{16\,h} \quad \text{(9)}$$

At the junction lines b the parabolas from the dome runs into similar parabolas in the hyperbolic paraboloid, the latter, however, having reversed curvature. If the thrust is assumed to remain unchanged these parabolas will have a downward vertical resultant along the entire length of the arch of the magnitude $w/2$. The intersecting set of arches within the hyperbolic parabolic area will therefore in addi-

This value for X_3 inserted in Eq. (7) gives:

$$\left(\frac{L}{2}+Y_3\right)^2+Y_3=\frac{L^2}{4}\left(1-\frac{Z}{h}\right)$$

Since $Y_3 = Y_2 - L/2$ it follows:

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With $L^2/4$ eliminated and the resulting expression divided by 2 and changing signs throughout it is found:

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This is, however, the same curve as established by using Eq. (6) and proves that the two surfaces are at the same elevation at any point along the junction line.

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$$F = H\cos w/\cos u \quad \text{.......... (8)}$$

where H is the unit horizontal thrust, u, the angle of slope of the shell surface in a direction normal to the horizontal projection of the section, and w is the slope of the section itself. The unit stress F would not necessarily be exactly acting normal to the section, but for small values of the angle w the discrepancy would be small and could safely be ignored in the design. For larger values of w an adjustment may be required. Eq. (8) is general and applicable to any type of thin shell structure, and the stress investigation is thereby reduced to the determination of the required horizontal thrust.

As shown in Example 1 the parabolic dome may be assumed to be divided into two series of intersecting arches by vertical planes running parallel with the junction lines. These arches so established will all be identical, a fact readily verified by the knowledge that a parabolic dome may be described by translating one parabola along and normal to a similar parabola. Since the height of these parabolas is $h/2$ and the span of the same parabolas is equal to $L/\sqrt{2}$ and further since the load, due to absolute symmetry of the shell in this area, may be assumed divided equally between the two sets of arches, the required thrust throughout and along the edges of the dome area is determined to be:

$$H_1=\frac{wL^2}{16h} \quad \text{.......... (9)}$$

At the junction lines b the parabolas from the dome runs into similar parabolas in the hyperbolic paraboloid, the latter, however, having reversed curvature. If the thrust is assumed to remain unchanged these parabolas will have a downward vertical resultant along the entire length of the arch of the magnitude $w/2$. The intersecting set of arches within the hyperbolic parabolic area will therefore in addition to the load w have to carry the load $w/2$ from the first set of arches. The later group is composed of arches with concave curved parabolas similar to those in the dome area. The required thrust to carry the total load of $3/2\,w$ is therefore established as:

$$H_2 = \frac{3\,wL^2}{16\,h} \qquad (10)$$

The compatability requirements between parallel running arches with different thrust values has been found to be of no serious consequence. It may be stated here, without going further into this problem, that instead of a sudden change in thrust from one group to the adjacent, it may be assumed that a gradual transition will take place. Minor moments and shears normal to the surface of the shell do accordingly exist along the junction lines b.

The established thrust of [Eq. (10)] carries all load down to the lines a at the junction with Area 1a.

The triangular Areas 1a limited by the lines a and bordering Area 1 form somewhat of a transition surface as far as the stresses are concerned. These stresses are not as readily determined as in the other parts of the shell. It may be concluded, however, that due to its similarity with the elastic membrane, the stresses in these areas will be governed by the same basic stress pattern as the rest of the shell. It is important to note, however, that due to the convex curvature of both sets of arches in these areas a certain amount of shear exists. The complete picture of what happens in the Area 1a can only be arrived at after the Area 1 has been investigated.

It was stated earlier that Area 1 could take a number of shapes. According to Eq. (5) they all must be rotation symmetrical, however, and the slope of the surface along the junction with the Areas 1a and 2 must coincide with those of the latter areas. This establishes the required thrust along the border of Area 1, this border being a circle with a radius of $R = L/\sqrt{8}$. The slope angle along this border is given from Eq. (6) for $X_2 = -L/4$ and $Y_2 = L/4$ as $\tan U = Z_2' = 8h/L^2$ $(Y_2 \cos \phi + X_2 \sin \phi)$ with $\phi = 135$ deg.
Since $\sin 135 \text{ deg} = 1/\sqrt{2}$ and $\cos 135 \text{ deg} = 1/\sqrt{2}$ it is:

$$Z_2' = -\frac{8h}{L^2}\left(\frac{L}{4\sqrt{2}} + \frac{L}{4\sqrt{2}}\right) = -\frac{\sqrt{8}\,h}{L}$$

The total vertical load to be carried by this circle may be established from a free body diagram separating Area 1 from the rest of the shell:

$$P_{x = L/\sqrt{8}} = 2wL^2 - \frac{\pi w L^2}{8}$$

Since the vertical component of the stress is equal to the horizontal thrust H_3 times Z_2', the following expression is established for this thrust:

$$H_3 = \frac{wL^2\,(2 - \pi/8)}{2\pi h} \qquad (11)$$

This thrust, which also may be expressed as $H_3 = 0.255\ wL^2/h$ as compared with $H_2 = (3wL^2)/16h$ or $H_2 = 0.188\ wL^2/h$ along the junction lines a, explains the existence of shears in the Area 1a.

The last Area 1 may now be fully investigated. Several approaches are possible, the most logical one appears to be to assume that the established thrust of Eq. (11) is maintained throughout this area. Based upon this assumption the following differential equation may be derived:

$$Z' = \frac{\left(2 - \pi\left(\frac{X}{L}\right)^2\right)h}{(2 - \pi/8)\,X}$$

which gives as general solution:

$$Z = 1.244\ h \log_e (X/L) - 0.977\ h\ (X/L)^2 + C_1$$

Inserting the edge requirement that for $X = L/\sqrt{8}$; $Z = -h/2$ the constant C_1 may be determined:

$$C_1 = -h/2 - 1.244\,h \log_e (1/\sqrt{8}) + 0.977\,h\,(1/8) = 0.916\,h \text{ and it follows:}$$

$$Z = 1.244\,h \log_e (X/L) - 0.977\,h\,(X/L)^2 + 0.916\,h \qquad (12)$$

This result indicates a logarithmic radial curve for the funnel surface. The extreme values of Z for X approaching zero may be disregarded since the shell in this vicinity will have fused into the column.

With the entire surface determined as also is the corresponding horizontal thrust, the actual stresses are readily determinable by means of Eq. (8).

EDGE LOADS ON THE STRUCTURAL MEMBRANE

This discussion is limited to square, structural membranes with straight horizontal edges. This divides the previously discussed surface into two types each with a side dimension of L as shown with dotted lines in Fig. 11. The one containing the column is referred to as "the column unit;" the other as "the dome unit." In the following a precast structure utilizing these two types of units is considered. In this case all areas adjacent to the edges are hyperbolic paraboloids. The problem of edge loads on these structural membranes is therefore similar to that of the hyperbolic paraboloid. A basic difference exists, however, in that in a conventionally designed hyperbolic paraboloid its support is so selected that only shears may be assumed to exist along its edges, whereas in the structural membrane these edges are subjected to large normal forces in addition to the shears.

The following is a general approach to the solution of this problem. The horizontal edges of the structural membrane require two types of edge loads to maintain equilibrium within the shell surface, one tangential force U acting normal to the edges and one shear force S acting along the edges. The force U may be divided into two components, one component V vertical and one component H horizontal. Since the horizontal thrusts, H_1 and H_2, as earlier established, describe the entire horizontal load on the shell, these thrusts may be used to determine the horizontal edge load H. The stresses H_1 and H_2 may be found to act in two directions inclined 45 and 135 deg to the edges, and the horizontal edge load is therefore:

$$H = \frac{WL^2}{16\,h}(3 \sin 45 \text{ deg} + 1 \sin 135 \text{ deg})\ ½ = \frac{wL^2}{8\,h} \text{.........} (13)$$

or also $H = 2\,H_1$.

This force is constant along the entire edge. The vertical edge load v is then determined as H tan u, where u is the slope of the surface normal to the edge. This tan u varies linearly from zero at the corner to a maximum value tan $u = 4h/L$ at the center of each side. The total vertical load on the unit is therefore:

$$W = 4LH\,(\tan u)_{max}\,1/2 = 4L\,\frac{wL^2}{8h}\,\frac{4h}{L}\ 1/2 = wL^2$$

This force acts upward on the dome unit and downward on the column unit and checks with the over-all statical requirements of the structural membrane.

The shear s may similarly be established from the two thrust values H_1 and H_2. It is:

$$S = \frac{wL^2}{16h}(3 \cos 45 \text{ deg} + 1 \cos 135 \text{ deg})\ 1/2 = \frac{wL^2}{16h} \text{...........} (14)$$

or also $S = H_1$.

This shear is constant throughout. On the dome unit it acts from the midpoint of each side toward the corners and in opposite direction on the column unit.

For the free edges of any larger structural membrane it is recommended that the vertical edge loads be transferred from the dome unit to the column unit by means of vertically placed edge beams. (A 10 ft square test unit 2 in. thick, however, showed negligible deflections of the edge when loaded to 300 lb along one unsupported edge.) Within the structure itself this same vertical edge load is automatically cancelled by the fact that each dome unit rests on the edges of the column-unit. The horizontal edge loads do not normally cancel out, however. Only if mechanical fastening devices were provided at the junction between the units a smaller or larger part of these horizontal forces would cancel out within the structure. In the following no such devices will be assumed. These horizontal loads would set up tension in the structural membrane.

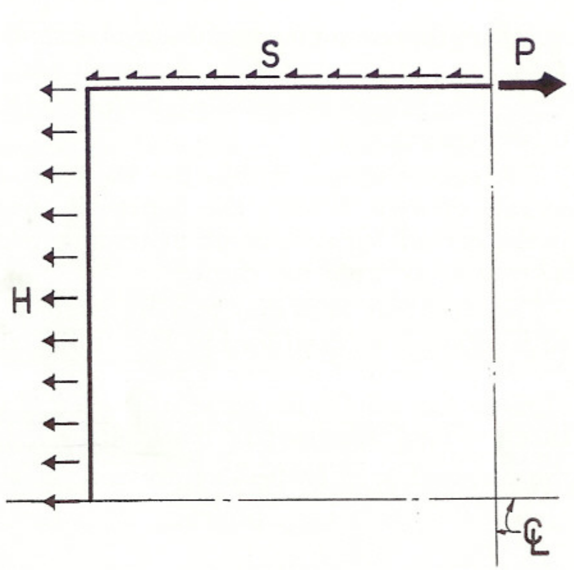

Fig. 13 — Horizontal edge loads in direction of *x*-axis. One-quarter of structural membrane column unit is shown

If prestressing wires are used to balance the horizontal loads, the magnitude of the required prestressing may be determined from the free body diagram in Fig. 13. Here one quarter of the column unit has been shown. Only the forces in the direction of the X-axis have been indicated. The forces in the Y-axis would be the same due to the symmetry of the unit. The diagram establishes the following values for the required prestressing force:

$$P_{reqd} = H\,L/2 = 2H_1\,L/2 = H_1L \text{ at the corners}$$

and

$$P^{max}{}_{reqd} = H_1L + S\,L/2 = 3/2\ H_1L \text{ at the center line of the column unit.}$$

The corresponding values for the dome unit are:

$$P^{max}{}_{reqd} = H_1L \text{ at the corners}$$

and

$$P_{reqd} = H_1L - S\,L/2 = 1/2\ H_1L \text{ at the center line of the dome unit}$$

Whereas this prestressing force satisfies all of the over-all statical requirements of the shell the combined effect of the edge loads and the prestressing force on the internal stresses in the structural membrane must be analyzed.

The following discussion presents a more qualitative than quantitative evaluation of this effect, any exact analysis not being available at the time this paper is written. In the author's opinion, however, this discussion comes close to describing the actual stress pattern within the shell.

Fig. 14 represents one quadrant of a unit. The combined prestressing force from the two sides is shown to have a diagonal resultant $P\sqrt{2}$ at the corner. The shears along the prestressing wire have been ignored in this diagram.

The curves shown within the shell are contour lines plotted at intervals of $h/10$. Within the hyperbolic paraboloidal part of the unit these are all hyperbolas of increasing focal distances as they occur further away from the corner.

These contour lines may now be assumed to be separate, rigid arches each carrying an axial load p. Due to the curvature of the axis of these arches horizontal reactions h are created. The approximate distribution of these reactions is indicated in Fig. 14. If $\Sigma p = P$, it is readily proven that the total resultant of these forces h will be in equilibrium with the diagonal load $P\sqrt{2}$. The presence of shears and forces normal to the horizontal arches will modify but not basically change this stress pattern.

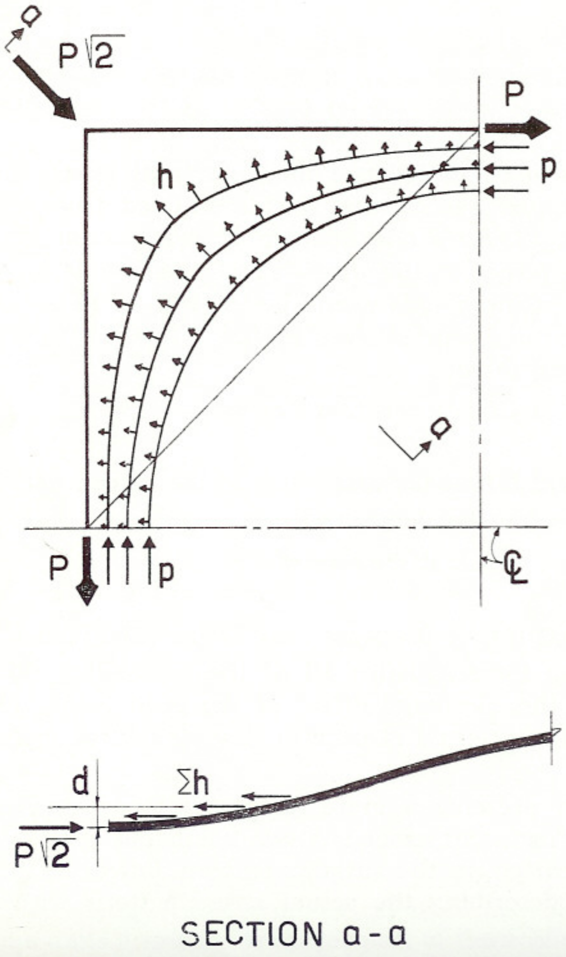

Fig. 14—Stresses due to prestressing. One-quarter of structural membrane unit is shown

Since these arches by definition are horizontal, the compressive forces p have no vertical component. Accordingly the basic equilibrium of the shell in the vertical direction has not been disturbed.

The horizontal edge loads H for a dome unit are shown in a similar diagram in Fig. 15. The contour lines are the same horizontal, hyperbolic arches, this time each subjected to an axial tension p. Again no vertical reactions result from this stress. The approximate distribution of the required horizontal reactions h have been indicated. As in the first loading, shears and forces normal to the arch axis must exist to complete the detailed equilibrium in the shell, and again these would modify the main stresses shown. If Σp is assumed equal to $HL/2 = P$, it is evident that the resultants from all reactions h of all arches are in equilibrium with the total edge load.

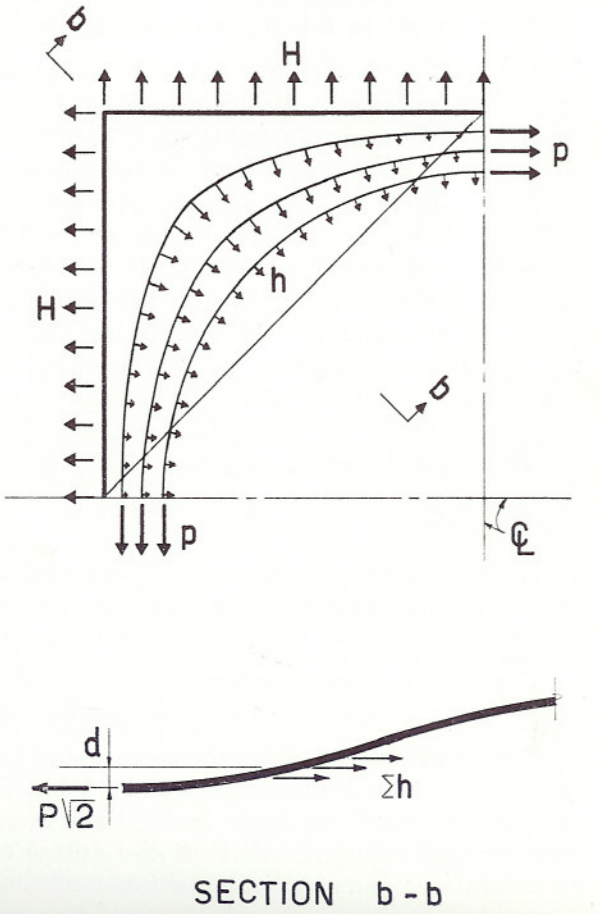

Fig. 15—Stresses due to horizontal edge loads. One-quarter of structural membrane unit is shown

By superimposing the two loading cases it becomes apparent that a greater part of the axial loads and their reactions will cancel out. It may be shown, however, that the shears from the two same loading cases will add together. Similarly, the heavy compression normal to the arch axis directly under the load $P\sqrt{2}$ will basically remain as in the first loading case.

Section a-a in Fig. 14 shows the resultant prestressing force $P\sqrt{2}$ and the horizontal reactions from the hyperbolic arches. Because these forces occur at various levels on the shell surface, radial moments are created. If the distance d between the location of the prestressing force and the resultant of the arch reactions were known, the total maximum radial moment could be determined as $M_r^{max} = P\sqrt{2}\, d$.

Section b-b in Fig. 15 shows the similar condition for the other loading case. A combination of the two loadings indicate that the greater part of the radial moments would cancel out. A crude estimate of the maximum radial moment due to the prestressing alone will be shown in the following example.

For a 20 x 20-ft dome unit with a height $h = 2$ ft a required prestressing force of 20,000 lb is assumed. If the maximum concrete stress is to be within an allowable limit of $f_c^{max} = 2000$ psi and assuming a rectangular stress block, the required concrete area to carry this load is $20/2 = 10$ sq in. With a shell thickness assumed to be 2 in. this stress block would have to extend 5 in. in from the edge with its resultant located a minimum distance of 2½ in. away from the edge of the unit. It is apparent from Fig. 14 that the heaviest stress concentration occurs near the midpoint of each side of the unit. The slope of the shell normal to the edge at this location is $4h/L = 0.4$ and the distance d is therefore $d = 0.4 \times 2\frac{1}{2} = 1$ in. The maximum total radial moment is $M_r = 1 \times 20 \times 1.414 = 28.28$ in.-kips $= 2.35$ ft-kips. The distribution of this moment along the edge is not known. If uniform distribution over one-half of the total length is assumed, the radial moment $M_r = 0.24$ ft-kips per ft results.

If a triangular stress block is assumed in place of the rectangular block and if the maximum stress at the edge is as before limited to 2000 psi, the radial bending moment would be 32 percent larger.

The local high compressive stresses at the corners of the units would have to be handled as in ordinary prestressed concrete design.

As stated earlier much theoretical and practical research is needed before the edge load effects will be fully understood. It appears, however, that some similarity exists between this problem and that of a deep beam design, and it is the author's opinion that they are not substantially more critical than that of the corresponding deep beam.

POSITIVE PROPERTIES OF THE STRUCTURAL MEMBRANE

Typical for all structural membranes is that the surface generally contains two or more sets of straight, horizontal lines. This is important architecturally, because it permits easy joining of a series of repeating shapes to cover wide areas. In the structural membrane described in Example 3 a uniform column spacing was used with a resulting square outline of these repeating shapes.

Like all thin shell structures, structural membranes depend upon horizontal thrust for their carrying capacities. Unlike most other thin shells, barrel shells for example, structural membranes do not need tie rods outside the surface itself. The straight lines described above provide an ideal location for tie rods, in form of pretensioned or post-tensioned strands. This simplifies anchorage, fireproofing, and similar problems in design and construction.

Another important feature of the structural membranes is the simplicity of the main curves in this shape. A template made up of two parabolas, one convex and the other concave, with each a height of $h/2$ and a span of $L/\sqrt{2}$ and connected with each other, if translated in the diagonal direction and sliding along the previously described straight lines, determines completely the Areas 1a, 2, and 3. These areas make up more than 80 percent of the entire surface. The only remaining part of the shell, Area 1, may be determined by rotating a second screed around the column, this template to conform in shape to the radial curve of the funnel. These two properties of the structural membranes are in the author's opinion the reason why this shape may become one of the most economical types known.

The simplicity in layout will be appreciated by any architect or engineer familiar with the problems in thin shell design. The same simplicity in the field does away with the costly layout work commonly connected with thin shell construction. The ease of forming or creating templates for forming is shown in construction recommendations.

CONSTRUCTION, ERECTION, DRAINAGE, AND WATERPROOFING OF STRUCTURAL MEMBRANES

Besides the standard method of cast-in-place construction, the structural membrane lends itself ideally to two modern construction methods. The first method is to cast the shell on the ground with the earth shaped to the proper form. After proper curing of the concrete the shell may then be stressed along the lines of the prestressing wires. Since all of its load is taken by column supports with no interferring tie rods at the support line it may now be lifted in place by a method similar to the lift slab method of erection.

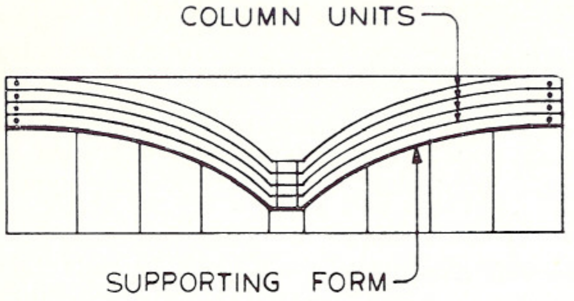

Fig. 16a—Precasting of units. Constant vertical thickness permits elements to be nested on jobsite casting beds

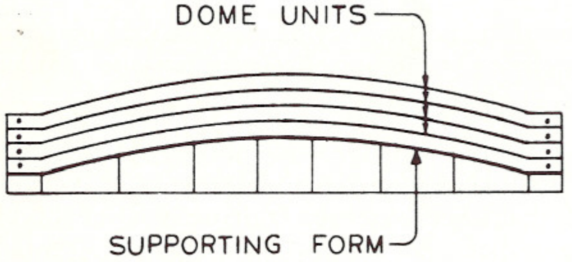

Fig. 16b—Casting arrangement for dome units

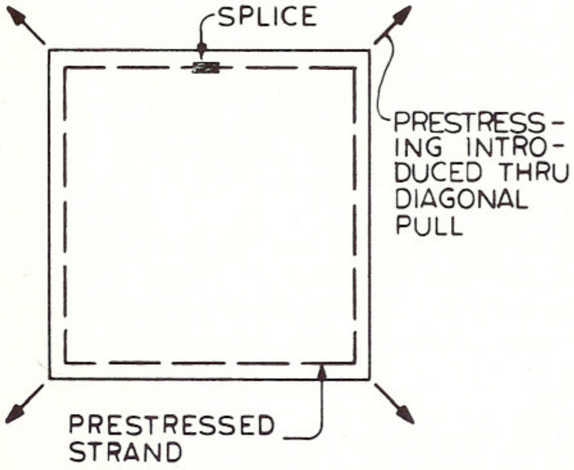

Fig. 17 — Arrangement of prestressing strand

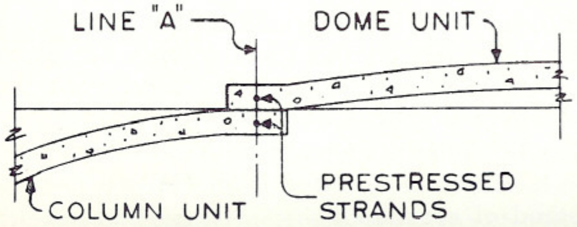

Fig. 18—Section through a construction joint

The second method of construction would primarily apply to structures with spans up to 40 or 50 ft with possibly 30 ft as the most economical. For this type the shells may be precast on a job site casting bed. The units will be of two types, referred to as the column units and the dome units. They will both have square outlines and the general dimension, L, of the units will be equal to $S/\sqrt{2}$, with S the column spacing in the structure. Due to the uniform stress typical for the structural membranes these units may be designed to have constant vertical thickness and therefore nested in the precasting bed (Fig. 16a and 16b).

The prestressing of each unit makes structural membrane units independent of the adjacent one for their load carrying capacity (Fig. 17). A series of simple construction joints are therefore possible. The typical one is shown in Fig. 18 with one unit merely resting on top of the edge of the other. Epoxy resins or other sealants placed in these joints would in most cases make them watertight. Mechanical clamping at the corners prevents the units from moving relative to each other. The column units are set on a flange or a seat on the column tops and cast-in-place concrete around the extension of the column provides a stiff moment connection between the column and the column unit.

The handling of the units during erection is simplified by the integral stability of each unit. The need for strong backs of vacuum pads is thereby eliminated. Placed in the building only the column unit requires light temporary shoring until the cast-in-place concrete has set up.

The over-all structural system is extremely stable for lateral loads as it consists of two intersecting series of three-hinged arches.

With carefully calked joints the roof should be watertight without any built-up roofing. The prestressing and the presence of only compressive stresses would eliminate the formation of cracks in the shell surface and consequently only a sprayed-on sealer would be required to prevent water from penetrating the shell.

Due to the shape, each column requires a down spout and the plumbing of the roof would normally cost more than in conventional design. Plastic tubing and fittings might offset some of this disadvantage and provide very efficient maintenance-free drainage.

CONCLUSION

The limitations of the use of a structural membrane are evident from its appearance. The surface must be curved. As with all thin shell work, this does not lend itself to multistory construction. A certain limitation as to selection of column spacing and arrangement exists. The problem of forming and finishing is more involved than the same for flat plates, ribbed or waffle slabs, and many other flat concrete floors or roofs.